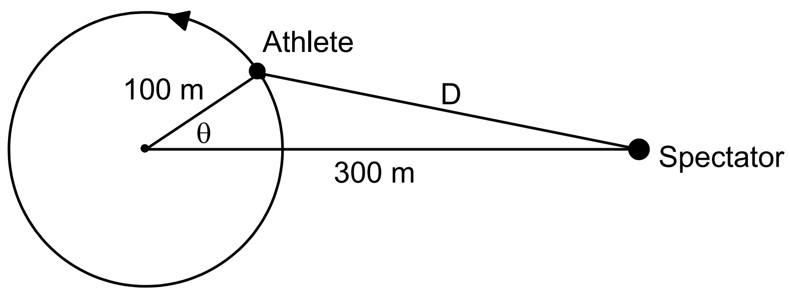
MORE PRACTICE

Question One

An athlete is running around a circular track of radius 100 *m* at 5 ms^{-1}. A spectator is 300 *m* from the centre of the track.



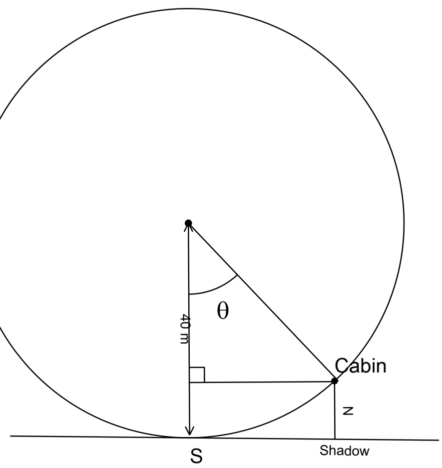
(a) Show that $\dfrac{d\theta}{dt} = 0.05\, rad\ s^{-1}$. [1]

(b) How fast is the distance (D) between them changing when the runner is running away from the spectator and the distance between them is 250 m? HINT: Use the Cosine Rule. [5]

Question Two

(11 marks)

Laura is travelling on a Ferris wheel of radius 40 metres, that is that is turning at a constant angular speed of one revolution every 5 minutes. Initially, Laura's cabin is at ground level, at point S.



(a) Find the rate as an exact value, in metres per minute, at which the height of Laura's cabin is increasing when it is 20 metres above the ground. (6 marks)

(b) If x metres is the displacement of the shadow from point S on the ground, obtain an expression for x in terms of t, where t is time in minutes. By finding as expression for the acceleration in terms of x, show that the shadow moves with simple harmonic motion.

(3 marks)

(c) The sun is directly overhead and casts a shadow on the ground directly below Laura's cabin. Find the speed at which the shadow is moving horizontally, when the cabin is 20 metres above the ground.

(2 marks)

Question Three

The shape of a vase is modelled by rotating the function

$$f(x)=\begin{cases} 5\sin\left(\dfrac{x}{10}\right)+3 & 0\le x\le a \\ 3 & a<x\le 40 \end{cases}$$

about the x-axis (x is in cm).

The function is graphed below.

Determine:

(i) the exact value of a, [2]

(ii) the volume of the vase (to the nearest cm^2). [3]

Question Four

Determine

(a) $\int \frac{3\tan^2 x}{\cos^2 x}\,dx$

(b) $\int_0^1 \frac{6x^2}{x^3+1}\,dx$

Question Five. **[12 marks]**

Two variable resistors with resistance *M* Ohms and *N* Ohms respectively are connected in parallel so that the Total Resistance *R* Ohms is given by $\frac{1}{R} = \frac{1}{M} + \frac{1}{N}$.

(a) Use implicit differentiation to write a differential equation linking $\frac{dR}{dt}$, $\frac{dM}{dt}$ and $\frac{dN}{dt}$ [3]

(b) At the instant when M = 50 Ohms and N = 200 Ohms, M is increasing at a rate of 10 Ohms per minute.

(i) Find R at this instant. [1]

(ii) Use Calculus methods to determine the rate of change of N (in Ohms per

minute), at this instant, if R is increasing at a rate of 5 Ohms per minute.
Show clearly how you obtained your answer.

(c) Given that $M = N^2$, use the incremental formula to find the approximate change in R when N changes from 50 Ohms to 51 Ohms.

Question Six [4 marks]

A curve is defined implicitly by: $\dfrac{\sin y}{x-1} = ay$, where a is a constant.

Find the value of the constant a in the curve if the tangent to the curve at point $(2, \dfrac{\pi}{3})$ is given by the equation $2x + 3y = \dfrac{\pi}{4}$.

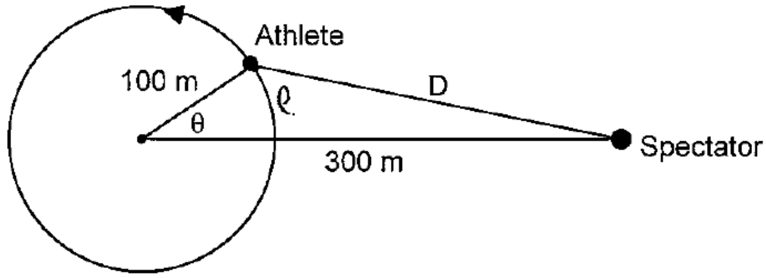
Leave your answer as an exact value.

Question Seven 14. [5 maks]

A cameraman is asked to film an aircraft flying overhead for a dramatic scene in a movie. The cameraman fixes his camera at ground level and rotates the camera in a vertical arc as the aircraft moves towards him. The aircraft approaches the cameraman flying at a speed of 750 *km/h* whilst maintaining a constant height of 8000 *m* above the ground. At what rate is the camera rotating (units in degrees/second) when the horizontal distance of the aircraft is 5 *km* from the cameraman.

Question One : SOLUTION

(6 marks)
An athlete is running around a circular track of radius 100 *m* at 5 ms^{-1}. A spectator is 300 *m* from the centre of the track.



(a) Show that $\frac{d\theta}{dt} = 0.05 \ rad \ s^{-1}$. [1]

In 1 sec $l = 5$

$l = 100\theta$ ✓ $\therefore \frac{d\theta}{dt} = 0.05 \text{ rad s}^{-1}$

$\theta = \frac{5}{100} = 0.05$

(b) How fast is the distance (D) between them changing when the runner is running away from the spectator and the distance between them is 250 m?
HINT: Use the Cosine Rule. [5]

Find $\frac{dD}{dt}$ at $D = 250$ $\theta = 0.89566$ ✓

$D^2 = 100^2 + 300^2 - 2(100)(300)\cos\theta$ ✓

$D = (100{,}000 - 60000\cos\theta)^{1/2}$

$\frac{dD}{d\theta}(100000 - 60000\cos\theta)^{1/2} \Big|_{\theta = 0.89566}$

$= 93.67$ ✓

$\frac{dD}{dt} = \frac{dD}{d\theta} \times \frac{d\theta}{dt}$ ✓

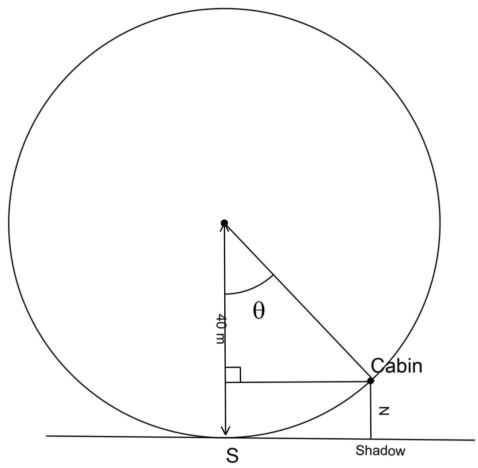
$= 93.67 \times 0.05$

$= 4.68 \text{ ms}^{-1}$ ✓

Question Two

(11 marks)

Laura is travelling on a Ferris wheel of radius 40 metres, that is that is turning at a constant angular speed of one revolution every 5 minutes. Initially, Laura's cabin is at ground level, at point S.



(a) Find the rate as an exact value, in metres per minute, at which the height of Laura's cabin is increasing when it is 20 metres above the ground. (6 marks)

Solution		
Let the radius joining the cabin to the centre of the wheel make an angle θ with the vertical. Wheel rotates every 5 minutes $\Rightarrow \dfrac{d\theta}{dt}=\dfrac{2\pi}{5}$ radians per minute The distance of the cabin below the centre of the wheel is $40\cos\theta$, Hence the distance above the ground $z=40(1-\cos\theta)$ $\Rightarrow \dfrac{dz}{dt}=\dfrac{dz}{d\theta}.\dfrac{d\theta}{dt}=40\sin\theta\dfrac{d\theta}{dt}$ $z=20 \Rightarrow \cos\theta=\dfrac{1}{2} \Rightarrow \sin\theta=\dfrac{\sqrt{3}}{2}$ $\dfrac{dz}{dt}=40.\dfrac{\sqrt{3}}{2}.\dfrac{2\pi}{5}=8\sqrt{3}\pi \approx 43.53$ metres per minute		
Specific behaviours	Mark allocation	Item classification
defines appropriate variables	1	complex
identifies rate of change of ϑ	1	complex
defines height above ground in terms of ϑ	1	complex
calculates $\sin\vartheta$ for z = 20	1	complex
uses chain rule to calculate appropriate rate of change	1	complex
accurately carries through calculation	1	complex

(b). If x metres is the displacement of the shadow from point S on the ground, obtain an expression for x in terms of t, where t is time in minutes. By finding as expression for the acceleration in terms of x, show that the shadow moves with simple harmonic motion.

(3 marks)

Solution

$$\frac{d\theta}{dt} = \frac{2\pi}{5} \Rightarrow \int d\theta = \int \frac{2\pi}{5}$$

$\theta = \frac{2\pi}{5}t + c$ and $\theta = 0$ and $t = 0 \Rightarrow c = 0$

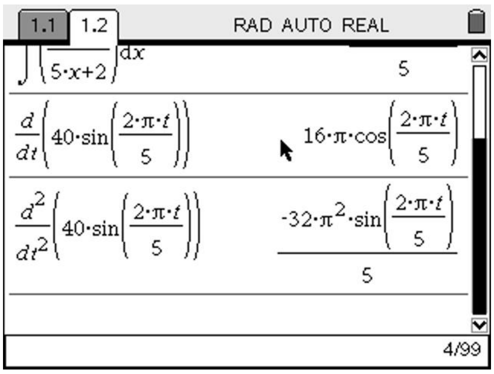
$$x = 40\sin\theta \Rightarrow x = 40\sin\frac{2\pi}{5}t$$

$$\frac{dx}{dt} = \frac{2\pi}{5} \times 40\cos\frac{2\pi}{5}t$$

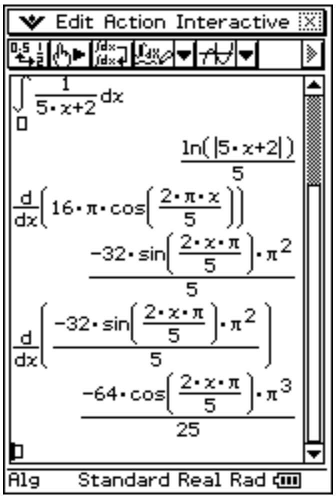
$$\frac{d^2x}{dt^2} = -\left(\frac{2\pi}{5}\right)^2 \times 40\sin\frac{2\pi}{5}t$$

$\frac{d^2x}{dt^2} = -k^2x$ which implies simple harmonic motion.

Using CAS:



or



Specific behaviours	Mark allocation	Item classification
uses $\frac{d\theta}{dt}$ to express ϑ in terms of *t*	1	complex
uses equation in (b) to define *x(t)*	1	complex
identifies acceleration equation using 2nd derivative	1	complex

(c). The sun is directly overhead and casts a shadow on the ground directly below Laura's cabin. Find the speed at which the shadow is moving horizontally, when the cabin is 20 metres above the ground. (2 marks)

Solution		
If metres denotes the distance of the shadow from S then; $x = 40\sin\theta$ $\frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = 40\cos\theta\frac{d\theta}{dt}$, when $z = 20 \Rightarrow \cos\theta = \frac{1}{2}$ also $\frac{d\theta}{dt} = \frac{2\pi}{5}$ $\frac{dx}{dt} = 40\times\frac{1}{2}\times\frac{2\pi}{5} = 8\pi$ m/min. Thus the shadow moves along the ground at a speed of 8π metres/minute.		
Specific behaviours	Mark allocation	Item classification
defines horizontal measurement in terms of ϑ calculates horizontal rate of change	1 1	complex complex

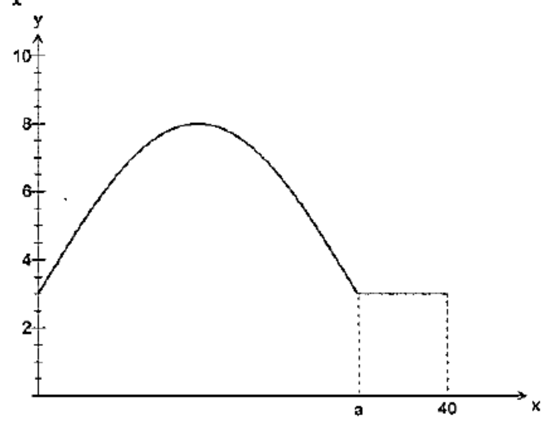
Question Three

(5 marks)
The shape of a vase is modelled by rotating the function

$$f(x) = \begin{cases} 5\sin\left(\frac{x}{10}\right) + 3 & 0 \le x \le a \\ 3 & a < x \le 40 \end{cases}$$

about the x-axis (x is in cm).

The function is graphed below.



Determine:

(i) the exact value of a, [2]

$5\sin\left(\frac{x}{10}\right)=0$ ✓

$\frac{x}{10}=0, \pi, \ldots$

$\therefore x=10\pi$ ✓

(ii) the volume of the vase (to the nearest cm^2). [3]

$V=\pi\int_0^{10\pi}\left[5\sin\left(\frac{x}{10}\right)+3\right]^2dx+\pi\int_{10\pi}^{40}3^2\,dx$ ✓ ✓

$=\pi(1275.44)+\pi(77.26)$

$=4249.63$

$\approx 4250\text{ cm}^2$ ✓

Question Four **(7 marks)**

(a) Determine $\int\frac{3\tan^2 x}{\cos^2 x}dx$ (2 marks)

Solution		
Substitute $u=\tan x$ and then $\frac{du}{dx}=\sec^2 x=\frac{1}{\cos^2 x}$ so $\int\frac{3\tan^2 x}{\cos^2 x}dx=\int 3u^2du=u^3+C$ So $\int\frac{3\tan^2 x}{\cos^2 x}dx=\tan^3 x+C$		
Specific behaviours	Mark allocation	Item classification
uses the chain rule	1	simple
carries through to correct solution	1	simple

(b) Evaluate $\int_0^1 \frac{6x^2}{x^3+1}\,dx$ (2 marks)

Solution		
Since $\frac{d}{dx}(x^3+1)=3x^2$ we see that $\int_0^1 \frac{6x^2}{x^3+1}\,dx = 2\int_0^1 \frac{3x^2}{x^3+1}\,dx$ $= 2\left[\ln(x^3+1)\right]_0^1$ $= 2(\ln 2 - \ln 1) = 2\ln 2$		
Specific behaviours	Mark allocation	Item classification
recognises form $\frac{f'(x)}{f(x)}$	1	simple
carries through to correct solution	1	simple

Question Five [12 marks]

Two variable resistors with resistance *M* Ohms and *N* Ohms respectively are connected in parallel so that the Total Resistance *R* Ohms is given by
$\frac{1}{R} = \frac{1}{M} + \frac{1}{N}$.

(a) Use implicit differentiation to write a differential equation linking $\frac{dR}{dt}$, $\frac{dM}{dt}$ and $\frac{dN}{dt}$ [3]

(a) $-\frac{1}{R^2}\frac{dR}{dT} = -\frac{1}{M^2}\frac{dM}{dT} - \frac{1}{N^2}\frac{dN}{dT}$ ✓✓✓ [3]

(b). At the instant when *M* = 50 Ohms and *N* = 200 Ohms, *M* is increasing at a rate of 10 Ohms per minute.

(i) Find *R* at this instant. [1]

(b) (i) *R* = 40 ✓ [1]

(ii) Use Calculus methods to determine the rate of change of *N* (in Ohms per

minute), at this instant, if R is increasing at a rate of 5 Ohms per minute.
Show clearly how you obtained your answer.

(ii) $-\frac{1}{40^2}\times 5 = -\frac{1}{50^2}\times 10 - \frac{1}{200^2}\times\frac{dN}{dT}$

✓

$\frac{dN}{dT} = -35$

✓

(c) Given that $M = N^2$, use the incremental formula to find the approximate change in R when N changes from 50 Ohms to 51 Ohms.

(c) $\frac{1}{R} = \frac{1}{N^2} + \frac{1}{N} \Rightarrow R = \frac{N^2}{N+1}$ ✓

Hence, $\frac{dR}{dN} = \frac{N^2+2N}{(N+1)^2}$ ✓✓

$\Rightarrow \delta R = \frac{N^2+2N}{(N+1)^2}\delta N$ ✓

When N = 50, $\delta N = 1 \Rightarrow \delta R = 0.9996$ Ohm ✓✓ [6]

Question Six

A curve is defined implicitly by: $\frac{\sin y}{x-1} = ay$, where a is a constant.

Find the value of the constant a in the curve if the tangent to the curve at point $(2, \frac{\pi}{3})$ is given by the equation $2x + 3y = \frac{\pi}{4}$.

Leave your answer as an exact value.

$2x + 3y = \frac{\pi}{4}$

$y = -\frac{2}{3}x + \frac{\pi}{12}$

$\frac{dy}{dx} = -\frac{2}{3}$ ✓ $\quad x = 2, \; y = \frac{\pi}{3}$

$\sin y = ay(x-1)$

✓ $\cos y \frac{dy}{dx} = a\frac{dy}{dx}(x-1) + ay$ ✓

$\cos\left(\frac{\pi}{3}\right)\left(-\frac{2}{3}\right) = a\left(-\frac{2}{3}\right)(2-1) + a\left(\frac{\pi}{3}\right)$ ✓

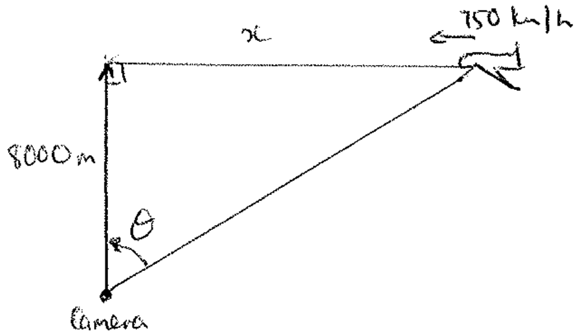
$\frac{1}{2} \cdot -\frac{2}{3} = -\frac{2}{3}a + \frac{\pi}{3}a$

$-\frac{1}{3} = a\left(\frac{\pi - 2}{3}\right)$

$a = \frac{1}{2-\pi}$ ✓

Question Seven

A cameraman is asked to film an aircraft flying overhead for a dramatic scene in a movie. The cameraman fixes his camera at ground level and rotates the camera in a vertical arc as the aircraft moves towards him. The aircraft approaches the cameraman flying at a speed of 750 *km/h* whilst maintaining a constant height of 8000 *m* above the ground. At what rate is the camera rotating (units in degrees/second) when the horizontal distance of the aircraft is 5 *km* from the cameraman.



$$\frac{dx}{dt} = 750 \text{ km/h} \quad ✓$$

$$\frac{d\theta}{dt} = ?$$

$$\theta = \tan^{-1}\left(\frac{5}{8}\right) \doteq 32° \quad ✓$$

$$\tan\theta = \frac{x}{8} \quad ✓$$

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{8}\frac{dx}{dt} \quad ✓$$

$$\frac{d\theta}{dt} = \frac{1}{8}\frac{dx}{dt}\cdot\cos^2\theta \quad ✓$$

$$= \frac{1}{8}(750)\cdot\cos^2(32)$$

$$= 1.873 \times 10^{-2} \text{ rad/sec}$$

$$= 1.07° / \text{sec} \quad ✓$$